Bernoulli

$$X \sim \text{Bernoulli}(p), \quad 0 \le p \le 1$$

$$f_X(x) = p^x(1-p)^{1-x}, \quad x \in \{0,1\}$$

$$\mathbb{E}[X] = p \qquad \text{Var}(X) = p(1-p)$$

Binomial

$$X \sim \text{Binomial}(n,p), \quad 0 \le p \le 1$$

$$X \overset{d}{\equiv} Z_1 + \cdots + Z_n, \quad Z_i \overset{iid}{\sim} \text{Bernoulli}(p)$$

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in \{0,1,\ldots,n\}$$

$$\mathbb{E}[X] = np \qquad \text{Var}(X) = np(1-p)$$

$$M_X(t) = (pe^t + q)^n, \quad q = 1-p$$

Multinomial

$$X \sim \text{Multi}(n, (p_1, \cdots, p_k)^T), \quad p_j \ge 0, \sum p_j = 1$$

$$f_X(x) = \binom{n}{x_1 \cdots x_k} p_1^{x_1} \cdots p_k^{x_k}$$

$$x_i = \{0, \cdots, n\}, \quad \sum x_i = n$$

Geometric

$$X \sim \text{Geo}(p), \quad 0 < p < 1$$

$$\text{Geo}(p) = \text{Negbin}(1,p)$$

$$f_X(x) = p(1-p)^{x-1}, \quad x \in \{1,2,\ldots\}$$

$$\mathbb{E}[X] = 1/p \qquad \text{Var}(X) = (1-p)/p^2$$

Negative Binomial

$$X \sim \text{Negbin}(r,p), \quad 0 < p < 1, r \in \mathbb{N}$$

$$X \overset{d}{\equiv} Z_1 + \cdots + Z_r, \quad Z_i \overset{iid}{\sim} \text{Geo}(p)$$

$$f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x \ge r, x \in \mathbb{N}$$

$$\mathbb{E}[X] = r/p \qquad \text{Var}(X) = r(1-p)/p^2$$

$$M_X(t) = (pe^t(1-qe^t)^{-1})^r, \quad t < -\log q$$

Poisson

$$X \sim \text{Poisson}(\lambda), \quad \lambda \ge 0$$

$$\text{Poisson}(\lambda) = \lim_{n\to\infty, np\to\lambda} \text{Binomial}(n,p)$$

$$f_X(x) = e^{-\lambda}\lambda^x/x!, \quad x \in \{0\} \cup \mathbb{N}$$

$$\mathbb{E}[X] = \lambda \qquad \text{Var}(X) = \lambda$$

$$M_X(t) = e^{-\lambda + \lambda e^t}$$

Exponential

$$X \sim \text{Exp}(1/\lambda), \quad \lambda > 0$$

$$\text{Exp}(1/\lambda) \overset{d}{\equiv} \text{Gamma}(1, 1/\lambda)$$

First arrival time for Poisson(λt)

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$F_X(x) = 1 - e^{-\lambda x}, \quad x > 0$$

$$F_X^{-1}(x) = -1/\lambda \log(1-x), \quad 0 < x < 1$$

$$\mathbb{E}[X] = 1/\lambda \qquad \text{Var}(X) = 1/\lambda^2$$

Gamma

$$X \sim \text{Gamma}(\alpha, \beta), \quad \alpha, \beta > 0$$

$$\alpha \in \mathbb{N}, X \overset{d}{\equiv} Z_1 + \cdots + Z_\alpha, \quad Z_i \overset{iid}{\sim} \text{Exp}(\beta)$$

Gamma($r, 1/\lambda$) is r^{th} arrival time for Poisson(λt)

$$X_i \overset{iid}{\sim} \text{Gamma}(\alpha_i, \beta), \sum X_i \sim \text{Gamma}\left(\sum \alpha_i, \beta\right)$$

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

$$\mathbb{E}[X] = \alpha\beta \qquad \text{Var}(X) = \alpha\beta^2$$

$$M_X(t) = (1-\beta t)^{-\alpha}, \quad t < 1/\beta$$

Gamma Function

$$\int_0^\infty x^{\alpha-1} e^{-x}\, dx \overset{\Delta}{=} \Gamma(\alpha), \quad \alpha > 0$$

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1) = 1$$

Normal

$$X \sim \text{N}(\mu, \sigma^2), \quad \sigma > 0$$

$$Z^2 \sim \chi^2(1), \quad Z = \frac{X-\mu}{\sigma} \sim \text{N}(0,1)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$\mathbb{E}[X] = \mu \qquad \text{Var}(X) = \sigma^2$$

$$M_X(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$$

Dirichlet

$$Y_1, \ldots, Y_k \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_{k+1})$$

$$X_1, \ldots, X_{k+1} \overset{iid}{\sim} \text{Gamma}(\alpha_i, \beta), \quad \alpha_i > 0$$

$$Y_i = \frac{X_i}{X_1 + \cdots + X_{k+1}}, \quad i = 1, \ldots, k$$

$$f_Y(y_1, \ldots, y_k)$$

$$= \frac{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{k+1})} \prod_{i=1}^k y_i^{\alpha_i - 1}$$

$$(1 - y_1 - \cdots y_k)^{\alpha_{k+1}-1} I_{(y_i > 0,\, y_1 + \cdots + y_k < 1)}$$

$$k = 1, \quad Y_1 \sim \text{Beta}(\alpha_1, \alpha_2)$$

Chi-square

$$X_1^2 + \cdots + X_r^2 \sim \chi^2(r), \quad X_i \overset{iid}{\sim} \text{N}(0,1), r > 0$$

$$\chi^2(r) = \text{Gamma}\left(\frac{r}{2}, 2\right)$$

Student's t

$$X = \frac{Z}{\sqrt{V/r}} \sim \text{t}(r), \quad r > 0$$

$$Z \sim \text{N}(0,1), V \sim \chi^2(r), Z \perp V$$

$$\text{t}(1) \overset{d}{\equiv} \text{Cauchy}(0,1)$$

$$f_X(x) = \frac{\Gamma\left((r+1)/2\right)}{\sqrt{r\pi}\,\Gamma\left(r/2\right)} \left(1 + x^2/r\right)^{-(r+1)/2}$$

Cauchy

$$X \sim \text{Cauchy}(\mu, \sigma), \quad \sigma > 0$$

$$= \mu + \sigma Z, \quad Z \sim \text{Cauchy}(0,1) \overset{d}{\equiv} \text{t}(1)$$

$$f_X(x) = \frac{1}{\pi\sigma}\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)^{-1}$$

Snedecor's F

$$X \sim \text{F}(r_1, r_2) = \frac{V_1/r_1}{V_2/r_2}$$

$$V_1 \sim \chi^2(r_1), V_2 \sim \chi^2(r_2), V_1 \perp V_2$$

$$F^{-1} \sim \text{F}(r_2, r_1), \quad \text{t}^2(r) \sim \text{F}(1, r)$$

$$f_X(x) = \frac{\Gamma\left((r_1+r_2)/2\right)}{\Gamma\left(r_1/2\right)\Gamma\left(r_2/2\right)} \left(\frac{r_1}{r_2}\right)^{r_1/2}$$

$$\frac{x^{r_1/2-1}}{(1 + r_1 x/r_2)^{(r_1+r_2)/2}} I_{(x>0)}$$

Fundamental Theorem of Normal Sampling

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$$

$$(i)\, \overline{X} \sim \mathcal{N}(\mu, \sigma^2/n)$$

$$(ii)\, s^2 = \frac{1}{n-1}\sum (X_i - \overline{X})^2 \perp \overline{X}$$

$$(iii)\, \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

Probability Integral Transformation

F is strictly increase, $F(X) \sim \text{U}(0,1)$, $F^{-1}(U) \overset{d}{\equiv} X$

$$U_{(r)} \overset{d}{\equiv} 1 - e^{-Z_{(r)}}, \quad Z_r \sim \text{Exp}(1)$$

Uniform Order Statistics

$$X_{(1)}, \ldots, X_{(n)} \sim \text{U}(0,1)$$

$$X_{(r)} \sim \text{Beta}(r, n-r+1)$$

$$(X_{(1)}, X_{(2)} - X_{(1)}, \ldots, X_{(n)} - X_{(n-1)})$$

$$\sim \text{Dirichlet}(1, \cdots, 1)$$

Exponential Order Statistics

$$X_{(1)}, \ldots, X_{(n)} \sim \text{Exp}(1), \quad Z_r \overset{iid}{\sim} \text{Exp}(1)$$

$$X_{(r)} \overset{d}{\equiv} \left(\frac{1}{n}Z_1 + \cdots + \frac{1}{n-r+1}Z_r\right)$$

Definition of Order Statistics

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} F(x), \quad h(y) = F^{-1}(1 - e^{-y})$$

$$(X_{(r)}) \overset{d}{=} (F^{-1}(U_{(r)})) \overset{d}{=} (F^{-1}(1 - e^{-Z_{(r)}})) \overset{d}{=} h(Z_{(r)})$$

Multivariate Normal Distribution

$$\boldsymbol{X} \overset{d}{=} \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = |2\pi\boldsymbol{\Sigma}|^{-n/2} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right)$$

$$m_{\boldsymbol{X}}(\boldsymbol{t}) = \exp(\boldsymbol{\mu}^T \boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T \boldsymbol{\Sigma} \boldsymbol{t})$$

$$\boldsymbol{AX} \perp \boldsymbol{BX} \iff \boldsymbol{A\Sigma B}^T = O$$

$$\boldsymbol{X_2} | \boldsymbol{X_1} = \boldsymbol{x_1}$$

$$\sim \mathcal{N}_q(\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{x_1} - \boldsymbol{\mu_1}), \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})$$

Definition of Chi-quare

$$\boldsymbol{z} \sim \mathcal{N}_k(\boldsymbol{0}, \boldsymbol{I}), \quad \boldsymbol{z}^T \boldsymbol{A} \boldsymbol{z} \sim \chi^2(r(\boldsymbol{A})) \iff \boldsymbol{A}^2 = \boldsymbol{A}$$

Convergence in Distribution

$$\lim_{n\to\infty} P(X_n \le x) = P(Z \le x) \quad \forall x : x \in \text{Conti}(F_Z)$$

$$\lim_{n\to\infty} P\left(\bigcap_i \boldsymbol{X}_{ni} \le \boldsymbol{x}_i\right) = P\left(\bigcap_i \boldsymbol{Z}_i \le \boldsymbol{x}_i\right)$$

$$\forall \boldsymbol{x} : \boldsymbol{x} \in \text{Conti}(F_{\boldsymbol{Z}})$$

Central Limit Theorem

$$X_1, \ldots, X_n \overset{iid}{\sim} (\mu, \sigma^2), \text{Var}(X_1) < \infty, Z \sim \mathcal{N}(0,1)$$

$$\frac{\sqrt{n}(\overline{X_n} - \mu)}{\sigma} \xrightarrow{d} Z \quad \forall x$$

$$\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \overset{iid}{\sim} (\boldsymbol{\mu}, \boldsymbol{\Sigma}), \text{Var}(\boldsymbol{X}_1) < \infty, \boldsymbol{Z} \sim \mathcal{N}_k(\boldsymbol{0}, \boldsymbol{\Sigma})$$

$$\lim_{n\to\infty} P(\sqrt{n}(\overline{\boldsymbol{X_n}} - \boldsymbol{\mu}) \le \boldsymbol{x}) = P(\boldsymbol{Z} \le \boldsymbol{x}) \quad \forall \boldsymbol{x}$$

Convergence in Probability

$$\lim_{n\to\infty} P(|X_n - X_\infty| > \epsilon) = 0 \quad \forall \epsilon > 0$$

$$\boldsymbol{X_n} \xrightarrow{p} \boldsymbol{c} \iff \lim_{n\to\infty} P(||\boldsymbol{X}_n - \boldsymbol{c}|| > \epsilon) = 0 \quad \forall \epsilon > 0$$

$$\iff \lim_{n\to\infty} P(\max|\boldsymbol{X}_{ni} - c_i| > \epsilon) = 0$$

$$\iff \boldsymbol{X}_{nj} \xrightarrow{p} c_j \quad \forall j$$

Inequalities

Markov: $X \ge 0, \quad P(X \ge k) \le \frac{1}{k}\mathbb{E}(X)$

Chevy: $\text{Var}(X) < \infty,\ P(|X - \mu| > k) \le \frac{1}{k^2}\text{Var}(X)$

Jensen: φ is convex, $\quad \varphi(\mathbb{E}(X)) \le \mathbb{E}(\varphi(X))$

$$\left|M_X(t)-\sum_{k=0}^{n}\frac{\mathbb{E}(X^k)t^k}{k!}\right|\leq\mathbb{E}\left[\min\left(\frac{|tX|^{n+1}}{(n+1)!},\frac{2|tX|^n}{n!}\right)\right]$$

Law of Large Numbers

$$X_1,\ldots,X_n\overset{iid}{\sim}\mathbb{E}(X_1)<\infty,\quad\overline{X_n}\overset{p}{\to}\mathbb{E}(X_1)$$

Delta Method

$$\sqrt{n}(\boldsymbol{X}_n-\boldsymbol{\theta})\xrightarrow[n\to\infty]{d}Z,\quad\dot{g}(\boldsymbol{\theta})\text{ is Continous}$$

$$\sqrt{n}(g(\boldsymbol{X}_n)-g(\boldsymbol{\theta}))\xrightarrow[n\to\infty]{d}(\dot{g}(\boldsymbol{\theta}))^TZ$$

$$\rho_4=\mathbb{E}\left[\left(\frac{X_1-\mu}{\sigma}\right)^4\right]-3$$

$$\sqrt{n}(s_n^2-\sigma^2)\overset{d}{\to}\mathcal{N}(0,(\rho_4+2)\sigma^4)$$

$$\sqrt{n}(s_n-\sigma)\overset{d}{\to}\mathcal{N}(0,(\rho_4+2)\sigma^2/4)$$