CH3. 단순선형회귀

$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(o, \sigma^2)$

$\hat\beta_0 + \hat\beta_1 \bar x = \bar y, \quad \hat\beta_0 \sum x_i + \hat\beta_1 \sum x_i^2 = \sum x_i y_i$

$\hat\beta_0 = \bar y - \hat\beta_1 \bar x, \quad \hat\beta_1 = \frac{s_{xy}}{s_{xx}}$

$e_i = y_i - \hat y_i, \quad \sum e_i = \sum x_i e_i = 0$

$$\begin{aligned}\text{SST} &= \sum (y_i - \bar y)^2 \\ &= \sum (y_i - \hat y_i)^2 + \sum (\hat y_i - \bar y)^2 \\ &= \text{SSE} + \text{SSR}\end{aligned}$$

$\text{SST} = s_{yy}, \quad \text{SSR} = \hat\beta_1^2 s_{xx}$

$R^2 = \frac{\text{SSR}}{\text{SST}} = \frac{\hat\beta_1^2 s_{xx}}{s_{yy}} = \frac{s_{xy}^2}{s_{xx} s_{yy}} = \gamma^2$

CH4. 단순회귀에 관한 추론

$T_\theta = \frac{\hat\theta - \theta_0}{\sqrt{\widehat{\text{Var}}(\hat\theta)}}, \quad \hat\theta \pm t_{\alpha/2}(\text{df}) \sqrt{\widehat{\text{Var}}(\hat\theta)}$

$\widehat{\sigma^2} = \frac{\text{SSE}}{n-2} = \text{MSE}, \quad \sqrt{\widehat{\sigma^2}/\sigma^2} \sim \chi^2(n-2)$

$\mathbb{E}(\text{SST}) = \beta_1^2 s_{xx} + (n-1)\sigma^2$

$\mathbb{E}(\text{SSR}) = \beta_1^2 s_{xx} + \sigma^2$

$\mathbb{E}(\text{SSE}) = (n-2)\sigma^2$

$\hat\beta_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{s_{xx}}\right)$

$\hat\beta_0 \sim \mathcal{N}\left(\beta_0, \sigma^2 \left(\frac{1}{n} + \frac{\bar x^2}{s_{xx}}\right)\right)$

$\hat\mu_{y\cdot x} \sim \mathcal{N}\left(\beta_0 + \beta_1 x, \sigma^2 \left(\frac{1}{n} + \frac{(x - \bar x)^2}{s_{xx}}\right)\right)$

$e_i \sim \mathcal{N}\left(0, \sigma^2 \left(1 - \frac{1}{n} - \frac{(x_i - \bar x)^2}{s_{xx}}\right)\right)$

CH5. 기초적 회귀분석의 기타 논제

(Box-Cox 변환)

$$z_i^{(\lambda)} = \begin{cases} \frac{y_i^\lambda - 1}{\lambda} \frac{1}{(\prod y_i^{\lambda-1})^{1/n}}, & \lambda \neq 0 \\ \log y_i \cdot (\prod y_i)^{1/n}, & \lambda = 0 \end{cases}$$

(측정오차)

$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad x_i^* = x_i + \delta_i$

$\delta_i \sim \mathcal{N}(0, \sigma_\delta^2) \perp \epsilon_i, \quad \text{data} : (y_i, x_i^*)$

Assumption : $\frac{1}{n} \sum (x_i - \bar x)^2 \xrightarrow{p} \eta_x^2$

$\hat\beta_1 = \frac{s_{x^*y}}{s_{x^*x^*}} \xrightarrow{p} \frac{\eta_x^2}{\eta_x^2 + \sigma_\delta^2} \beta_1$

(두 회귀선의 비교)

$y_{1j} = \beta_{10} + \beta_{11} x_{1j} + \epsilon_{1j}, \quad \epsilon_{1j} \sim \mathcal{N}(0, \sigma^2)$

$y_{2j} = \beta_{20} + \beta_{21} x_{2j} + \epsilon_{2j}, \quad \epsilon_{2j} \sim \mathcal{N}(0, \sigma^2)$

$(\text{H}_0 : \beta_{11} = \beta_{21})$

$\widehat{\sigma^2} = \frac{\text{SSE}_1 + \text{SSE}_2}{(n_1 - 2) + (n_2 - 2)}$

$\text{Var}(\hat\beta_{11} - \hat\beta_{21}) = \text{Var}(\hat\beta_{11}) + \text{Var}(\hat\beta_{21})$

$T_{\beta_1} = \frac{\hat\beta_{11} - \hat\beta_{21}}{\sqrt{\left(\frac{1}{s_{x_1x_1}} + \frac{1}{s_{x_2x_2}}\right)\widehat{\sigma^2}}} \sim t(n_1 + n_2 - 4)$

$(\text{H}_0 : \beta_{10} = \beta_{20}, \beta_{11} = \beta_{21})$

Reduced form : $y_{ij} = \alpha_0 + \alpha_1 x_{ij} + \epsilon_{ij}$

$\text{SSE}_F = \text{SSE}_1 + \text{SSE}_2, \quad \text{df}_F = n_1 + n_2 - 4$

$\text{SSE}_R = \sum_i \sum_j (y_{ij} - \hat\alpha_0 - \hat\alpha_1 x_{ij})^2,$

$\text{df}_R = n_1 + n_2 - 2$

$\text{F} = \frac{(\text{SSE}_R - \text{SSE}_F)/(\text{df}_R - \text{df}_F)}{\text{SSE}_F/\text{df}_F}$

$\overset{H_0}{\sim} \mathcal{F}(2, n_1 + n_2 - 4)$

Required

$(i)\ \frac{\text{SSE}_F}{\sigma^2} \sim \chi^2(\text{df}_F)$

$(ii)\ \frac{\text{SSE}_R - \text{SSE}_F}{\sigma^2} \sim \chi^2(\text{df}_R - \text{df}_F)$

$(iii)\ (\text{SSE}_R - \text{SSE}_F) \perp \text{SSE}_F$

(이차형식)

Under $\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{V}), A = A^T$

1. $Q = \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y} \sim \chi^2(\text{r}(\boldsymbol{A}), \frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{A} \boldsymbol{\mu})$

   $\Leftrightarrow \boldsymbol{AV}$ is idempotent.

2. $\mathbb{E}(\boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y}) = \text{tr}(\boldsymbol{AV}) + \boldsymbol{\mu}^T \boldsymbol{A} \boldsymbol{\mu}$

3. $\text{Cov}(\boldsymbol{y}, \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y}) = \boldsymbol{V}(\boldsymbol{A} + \boldsymbol{A}^T)\boldsymbol{\mu}$

4. $\boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y} \perp \boldsymbol{By} \Leftrightarrow \boldsymbol{BVA} = \boldsymbol{O}$

5. $\boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y} \perp \boldsymbol{y}^T \boldsymbol{B} \boldsymbol{y} \Leftrightarrow \boldsymbol{AVB} \vee \boldsymbol{BVA} = \boldsymbol{O}$

$\boldsymbol{A}_j, \boldsymbol{A} = \sum \boldsymbol{A}_j$ are symmetric

$$\begin{cases} \boldsymbol{y}^T \boldsymbol{A}_j \boldsymbol{y} \sim \chi^2(r(\boldsymbol{A}_j), \frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{A} \boldsymbol{\mu}) \\ \boldsymbol{y}^T \boldsymbol{A}_j \boldsymbol{y} \text{ are independent} \\ \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y} \sim \chi^2(r(\boldsymbol{A}), \frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{A} \boldsymbol{\mu}) \end{cases}$$

$$\Leftrightarrow \text{I} : \begin{cases} \text{Two of (a), (b), (c) hold} \\ \text{(a) } \boldsymbol{A}_j \boldsymbol{V} \text{ is idempotent} \\ \text{(b) } \forall i < j,\ \boldsymbol{A}_i \boldsymbol{V} \boldsymbol{A}_j = \boldsymbol{O} \\ \text{(c) } \boldsymbol{AV} \text{ is idempotent} \end{cases}$$

II : (c) and $\text{r}(\boldsymbol{A}) = \sum \text{r}(\boldsymbol{A}_j)$

(행렬의 미분)

$\frac{\partial(\boldsymbol{c}^T \boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{c}, \quad \frac{\partial(\boldsymbol{y}^T \boldsymbol{A} \boldsymbol{y})}{\partial \boldsymbol{y}} = (\boldsymbol{A} + \boldsymbol{A}^T)\boldsymbol{y}$

(최소제곱법)

$\boldsymbol{y}_{n\times 1} = \boldsymbol{X}_{n\times p'} \boldsymbol{\beta}_{p'\times 1} + \boldsymbol{\epsilon}_{n\times 1}$

$\boldsymbol{X}^T \boldsymbol{X} \hat{\boldsymbol{\beta}} = \boldsymbol{X}^T \boldsymbol{y}, \quad \hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{y}$

CH6. 중회귀분석 기초

$\hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}(\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{y} = \boldsymbol{Hy}$

$\boldsymbol{H}^2 = \boldsymbol{H}, \boldsymbol{H}^T = \boldsymbol{H}, \quad \boldsymbol{HX} = \boldsymbol{X}, \boldsymbol{H1} = \boldsymbol{H}$

$\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{\mu} = \boldsymbol{X\beta}, \sigma^2 \boldsymbol{I}_n)$

$\text{SST} = \boldsymbol{y}^T \left(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\right) \boldsymbol{y}$

$\frac{\text{SST}}{\sigma^2} \sim \chi^2 \left(n-1, \frac{1}{2}\boldsymbol{\mu}^T \left(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\right)\boldsymbol{\mu}\right)$

$\text{SSE} = \boldsymbol{y}^T (\boldsymbol{I}_n - \boldsymbol{H}) \boldsymbol{y}, \quad \frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-p-1)$

$\text{SSR} = \boldsymbol{y}^T \left(\boldsymbol{H} - \frac{1}{n}\boldsymbol{J}_n\right) \boldsymbol{y}$

$\frac{\text{SSR}}{\sigma^2} \sim \chi^2 \left(p, \frac{1}{2}\boldsymbol{\mu}^T \left(\boldsymbol{H} - \frac{1}{n}\boldsymbol{J}_n\right)\boldsymbol{\mu}\right)$

$\text{SSE} \perp \text{SSR}, \quad \frac{\text{MSR}}{\text{MSE}} \sim \mathcal{F}(p, n-p-1)$

$\hat{\boldsymbol{\beta}} \sim \mathcal{N}_n(\boldsymbol{\beta}, (\boldsymbol{X}^T \boldsymbol{X})^{-1}\sigma^2)$

(변수의 직교화)

$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} = \begin{bmatrix} \cdot & \boldsymbol{O} \\ \cdot & \cdot \end{bmatrix}$

$\underset{n\times 1}{\boldsymbol{y}} = \underset{n\times(q+1)}{\boldsymbol{X}_1} \underset{(q+1)\times 1}{\boldsymbol{\beta}_1} + \underset{n\times(p-q)}{\boldsymbol{X}_2} \underset{(p-q)\times 1}{\boldsymbol{\beta}_2}$

$\begin{bmatrix} \boldsymbol{X}_2^T \boldsymbol{X}_2 & \boldsymbol{X}_2^T \boldsymbol{X}_1 \\ \boldsymbol{X}_1^T \boldsymbol{X}_2 & \boldsymbol{X}_1^T \boldsymbol{X}_1 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\beta}}_2 \\ \hat{\boldsymbol{\beta}}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_2^T \boldsymbol{y} \\ \boldsymbol{X}_1^T \boldsymbol{y} \end{bmatrix}$

$\boldsymbol{X}_2^T (\boldsymbol{I}_n - \boldsymbol{H}_1) \boldsymbol{X}_2 \hat{\boldsymbol{\beta}}_2 = \boldsymbol{X}_2^T (\boldsymbol{I}_n - \boldsymbol{H}_1) \boldsymbol{y}$

$\boldsymbol{X}_2^T (\boldsymbol{I}_n - \boldsymbol{H}_1)^2 \boldsymbol{X}_2 \hat{\boldsymbol{\beta}}_2 = \boldsymbol{X}_2^T (\boldsymbol{I}_n - \boldsymbol{H}_1)^2 \boldsymbol{y}$

Let $\boldsymbol{X}_{2\cdot 1} = (\boldsymbol{I}_n - \boldsymbol{H}_1)\boldsymbol{X}_2$

$\boldsymbol{X}_{2\cdot 1}$은 $\boldsymbol{X}_2 = \boldsymbol{X}_1 \gamma_1 + \boldsymbol{\epsilon}$ 의 잔차와 동일

$\boldsymbol{X}_{2\cdot 1}^T \boldsymbol{X}_{2\cdot 1} \hat{\boldsymbol{\beta}}_2 = \boldsymbol{X}_{2\cdot 1}^T \boldsymbol{y}$

$\hat{\boldsymbol{\beta}}_2$은 $\boldsymbol{y} = \boldsymbol{X}_{2\cdot 1} \gamma_2 + \boldsymbol{\epsilon}$ 의 $\hat\gamma_2$과 동일

$\hat{\boldsymbol{\beta}}_2$은 $\boldsymbol{e} = \boldsymbol{X}_{2\cdot 1} \gamma_2 + \boldsymbol{\epsilon}$ 의 $\hat\gamma_2$과 동일

CH7. 중회귀분석의 추론

(Gauss-Markov 정리)

$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{y}$ is BLUE

Let $\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} + \boldsymbol{Ay}$, $\text{Var}(\tilde{\boldsymbol{\beta}}) - \text{Var}(\hat{\boldsymbol{\beta}}) = \sigma^2 \boldsymbol{A}\boldsymbol{A}^T$

$\text{Var}(\tilde{\boldsymbol{\beta}}) - \text{Var}(\hat{\boldsymbol{\beta}})$ positive definite

$\hat{\boldsymbol{\beta}}_j \sim \mathcal{N}_n(\boldsymbol{\beta}_j, ((\boldsymbol{X}^T \boldsymbol{X})^{-1})_{(j+1)\,(j+1)}\sigma^2)$

($\boldsymbol{C\beta} = \boldsymbol{m}$ 의 검정)

$\tilde{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}}{\arg\min} \{(\boldsymbol{y} - \boldsymbol{X\beta})^T (\boldsymbol{y} - \boldsymbol{X\beta}) \,|\, \boldsymbol{C\beta} = \boldsymbol{m}\}$

$\hat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}}{\arg\min} \{(\boldsymbol{y} - \boldsymbol{X\beta})^T (\boldsymbol{y} - \boldsymbol{X\beta})\}$

$\text{SSE}(\tilde{\boldsymbol{\beta}}) = \text{SSE}(\hat{\boldsymbol{\beta}}) + Q$

$Q = (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{m})^T (\boldsymbol{C}(\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{C}^T)^{-1} (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{m})$

$$\frac{Q}{\sigma^2} \sim \chi^2(r(\boldsymbol{C}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{C}^T) = k, \lambda)$$

$$\mathrm{SSE}(\hat{\boldsymbol{\beta}}) \sim \chi^2(n-p-1), \mathrm{SSE}(\hat{\boldsymbol{\beta}}) \perp Q$$

$$F_0 = \frac{Q/k}{\mathrm{SSE}(\hat{\boldsymbol{\beta}})/(n-p-1)} \sim \mathcal{F}(k, n-p-1, \lambda)$$

$$\sum_i \boldsymbol{e}_i = \sum_i \boldsymbol{x}_{ij}\boldsymbol{e}_i = \sum_i \hat{\boldsymbol{y}}_i\boldsymbol{e}_i = 0$$

CH8. 추정과 가설검정

$$SS(\hat{\beta}_0) = \boldsymbol{y}^T\frac{1}{n}\boldsymbol{J}_n\boldsymbol{y}$$

$$\hat{\boldsymbol{\beta}}_1 = (\beta_0, \dots \beta_q)^T, \hat{\boldsymbol{\beta}}_2 = (\beta_{q+1}, \dots \beta_p)^T$$

$$SS(\hat{\boldsymbol{\beta}}_2|\hat{\boldsymbol{\beta}}_1) = \boldsymbol{y}^T(\boldsymbol{H} - \boldsymbol{H}_1)\boldsymbol{y}$$

$$SS(\hat{\boldsymbol{\beta}}_1) = \boldsymbol{y}^T\boldsymbol{H}_1\boldsymbol{y}$$

$$(\boldsymbol{H} - \boldsymbol{H}_1)^2 = (\boldsymbol{H} - \boldsymbol{H}_1),\ r(\boldsymbol{H} - \boldsymbol{H}_1) = p - q$$

$$\frac{SS(\hat{\boldsymbol{\beta}}_2|\hat{\boldsymbol{\beta}}_1)}{\sigma^2} \sim \chi^2(p-q, \lambda)$$

$$\lambda = SS(\hat{\boldsymbol{\beta}}_2|\hat{\boldsymbol{\beta}}_1)\boldsymbol{\beta}_2^T\left[\boldsymbol{X}_2^T\boldsymbol{X}_2 - \boldsymbol{X}_2^T\boldsymbol{H}_1\boldsymbol{X}_2\right]\boldsymbol{\beta}_2/(2\sigma^2)$$

$$\mathrm{SSE} \perp SS(\hat{\boldsymbol{\beta}}_2|\hat{\boldsymbol{\beta}}_1)$$

(Joint Confidence Region)

$$\boldsymbol{y} = \beta_0 + \beta_1\boldsymbol{x}_1 + \cdots + \beta_p\boldsymbol{x}_p + \boldsymbol{\epsilon}$$

$$(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T(\boldsymbol{X}^T\boldsymbol{X})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma^2 \sim \chi^2(p+1)$$

$$\mathrm{SSE} \perp \hat{\boldsymbol{\beta}}$$

$$R = \left\{\boldsymbol{\beta} : \frac{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T(\boldsymbol{X}^T\boldsymbol{X})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})/(p+1)}{\mathrm{SSE}/(n-p-1)} \le F_\alpha(p+1, n-p-1)\right\}$$

(Simultaneous Confidence Interval)
1. Bonferroni Method

$$A_j = [\beta_j|\hat{\beta}_j \pm t_{\alpha/2(p+1)}\sqrt{\hat{\mathrm{Var}}(\hat{\beta}_j)}]$$

$$P\left(\bigcap_{j=0}^p A_j\right) = 1 - P\left(\bigcup_{j=0}^p A_j^c\right) \le \sum_{j=0}^p P(A_j) = \sum_{j=0}^p \frac{\alpha}{p+1} = \alpha$$